Stationary equations of non-autonomous symmetries

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A well-known construction scheme for exact solutions of integrable equations is related to stationary equations for higher symmetries from a commutative subalgebra. If we use also the noncommutative symmetries then the resulting reduction lose the Liouville integrability property, but retains the Painlevé property. Such reductions include string equations related to classical symmetries such as the Galilean boost and the scaling. These solutions are not very well studied, but we know from the works by Suleimanov, Dubrovin and others that among them there are special solutions that describe asymptotics in the vicinity of the breaking point.

In the talk, some recent results are presented that show that reductions obtained by use of master-symmetries may turn useful in studying another physically important regime, such as the decay of step-like solutions. Already published results are related to the KdV equation and the Volterra lattice; some new examples are also reported.

Outline of the talk

- General scheme
  - Commutative and noncommutative symmetries
  - Stationary equations
  - Lax representation
- Examples
  - KdV equation
  - Landau–Lifshitz equation
  - Volterra lattice

Motivation: it is likely that various scenarios of the development of oscillations in nonlinear models can be described by this kind of constraints

Commutative and noncommutative symmetries

Consider an evolution equation

$$u_t = f[u]$$

(either continuous PDE like KdV or DΔE like Volterra lattice) admitting a symmetry

$$u_T = g[u]$$

(that is, $[D_t, D_T] = 0$). Then the stationary equation

$$g[u] = 0$$

is a constraint consistent with the given equation. This is a well-known method to reduce the dimension of the problem, since we replace the original PDE with an ODE. What is the general form of such reduction? Recall that in a typical situation, an integrable hierarchy consists of two sequences of flows

$$u_{t_j} = f_j, \quad u_{\tau_j} = g_j, \quad j = 0, 1, 2, \dots$$

which constitute the Lie algebra with the commutation relations like

$$[D_{t_j}, D_{t_k}] = 0, \quad [D_{\tau_j}, D_{t_k}] = kD_{t_{j+k-1}}, \quad [D_{\tau_j}, D_{\tau_k}] = (k-j)D_{\tau_{j+k-1}}$$

(the picture can be more complicated for systems with several field variables).

Therefore, the general form of the stationary equation is

$$\alpha_0 f_0 + \cdots + \alpha_m f_m + \beta_0 g_0 + \cdots + \beta_n g_n = 0. \tag{1}$$

However, when we try to solve (1), it turns out that not all symmetries are equally good: their commutativity properties are important.

• *Novikov equation.* The most favourable case is when all $\beta_j = 0$, that is D_T belongs to a commutative Lie subalgebra of the symmetries. Then the stationary equation inherits this subalgebra, so that the ODE (1) turns out to be Liouville integrable. This is the case of finite-gap and multisoliton solutions.

• *Generalized "string" equation.* In contrast, if at least one $\beta_j \neq 0$, that is D_T contains some members of the additional noncommutative Lie subalgebra then the constraint (1) is not Liouville integrable. Instead, it is a Painlevé type equation. This is the case which we are interested in.

The noncommutative symmetries were discovered in

Ibragimov, Shabat, Dokl. Akad. Nauk SSSR **244:1** (1979) 57;
Fuchssteiner, Progr. Theor. Phys. **70:6** (1983) 1508;
Orlov, Shulman, Theor. Math. Phys. **64:2** (1985) 862;
Burtsev, Zakharov, Mikhailov, Theor. Math. Phys. **70:3** (1987) 227.

Lax representations

To explain the difference between D_{t_j} and D_{τ_j}, let us recall the construction of zero curvature repesentation with variable spectral paprameter from [BZM].

The auxiliary linear problems

$$\Psi_x = U\Psi, \quad \Psi_{t_j} = V_j\Psi, \quad \Psi_{\tau_j} + \lambda^j\Psi_\lambda = W\Psi \quad \Rightarrow$$

the compatibility conditions

$$U_{t_j} = V_{j,x} + [V_j, U], \qquad U_{\tau_j} + \lambda^j U_\lambda = W_{j,x} + [W_j, U].$$

Here U, V, W are matrices depending on the field variables and the spectral parameter λ. The commutation relation between D_{t_j} and D_{τ_k} are the same as for the operators

$$\lambda^j, \quad \lambda^k\frac{d}{d\lambda}.$$

If the stationary equation (1) contains a linear combination of f_j only then it admits the Lax representation

$$V_x = [U, V], \quad V = \sum \alpha_j V_j \qquad \Rightarrow \qquad D_x(\det V) = 0,$$

so that $\det V$ is the first integral for (1) depending on λ.

In the general case, the Lax representation for stationary equation takes the form which is usual in the method of isomonodromy deformations:

$$W_x = K(\lambda)U_\lambda + [U,W], \quad W = \sum(\alpha_j V_j + \beta_j W_j), \quad K = \sum \beta_j \lambda^j.$$

Now $\det W$ is preserved only at the zeroes of $K(\lambda)$:

$$D_x(\det W(\lambda_i)) = 0, \quad K(\lambda_i) = 0,$$

and this set of first intgerals is insufficient to provide the Liouville integrability.

Korteweg–de Vries equation

The Lie algebra of the KdV equation

$$u_t = u_3 + 6uu_1, \quad u_j := \partial_x^j(u),$$

is generated by the recursion operator $R = D_x^2 + 4u + 2u_1 D_x^{-1}$:

$$u_{t_j} = f_j = R^j(u_1), \quad u_{\tau_j} = g_j = R^j(6tu_1 + 1), \quad j = 0, 1, 2, \dots .$$

For instance,

$u_{t_0} = f_0 = u_1$	(x-translation)
$u_{t_1} = f_1 = (u_2 + 3u^2)_x$	(t-translation)
$u_{t_2} = f_2 = (u_4 + 10uu_2 + 5u_1^2 + 10u^3)_x$	(simplest higher symmetry)
$\dots$	
$u_{\tau_0} = g_0 = 6tu_1 + 1$	(Galilean boost)
$u_{\tau_1} = g_1 = 3tf_1 + xu_1 + 2u$	(scaling)
$u_{\tau_2} = g_2 = 3tf_2 + xf_1 + 4u_2 + 8u^2 + 2u_1u_{-1}$	(master-symmetry)
$\dots$	

We see that the noncommutative part of the hierarchy is also *nonautonomous* and *nonlocal*. In each flow, except for D_{τ_0} and D_{τ_1}, we have to add a new nonlocal variable which is equivalent to introducing a potential for the next conservation law. For instance, $v = u_{-1}$ in g_2 is differentiated according to the rule

$$v_x = u, \quad v_t = u_2 + 3u^2.$$

The stationary equation

$$E = \alpha_0 f_0 + \cdots + \alpha_m f_m + \beta_0 g_0 + \cdots + \beta_n g_n = 0$$

for any linear combination of the symmetries satisfies the identity

$$D_t(E) = (D_x^3 + 6uD_x + 6u_1)(E) = 0,$$

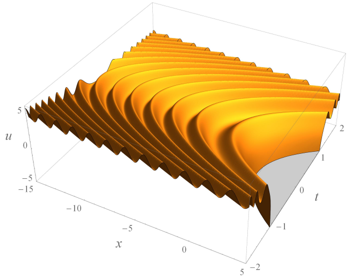
that is, this is some ODE consistent with KdV.

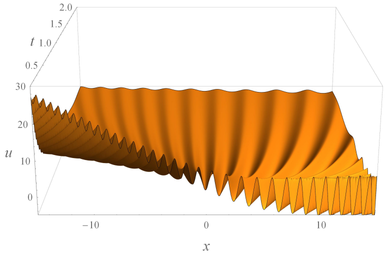
The case with all $\beta_j = 0$ brings to the construction of finite-gap solutions

Novikov, Funct. Anal. Appl. **8:3** (1974) 236;
Dubrovin, Matveev, Novikov, Russ. Math. Surv. **31:1** (1976) 59.

Adding of just one noncommutative symmetry g_j spoils the integrability, but retains the Painlevé property.

The stationary equation for the KdV itself + the Galilean or scaling symmetry is equivalent to the group reduction and results in the selfsimilar solutions governed by the P_1 and P_2 equations, respectively.





The next example is the so-called string equations which are of the form

$$\text{higher symmetry} \ + \ \text{classical symmetry} \ = 0$$

Moore, Comm. Math. Phys. **133:2** (1990) 261.

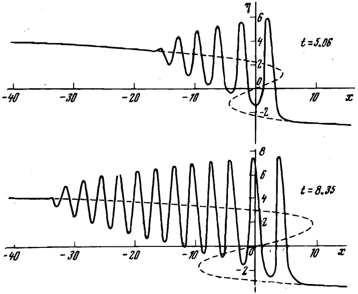
Oscillating zone in the vicinity of the breaking point

The string equations turn out to be important in the study of the Gurevich–Pitaevskii problem on the breaking of the wave front. The idea is very simple: take the stationary equation $f_2 + g_1 = 0$ and apply the *dispersionless limit*:

$$u_4 + 10uu_2 + 5u_1^2 + 10u^3 + 6tu + x = 0 \quad (2)$$
$$\downarrow$$
$$10u^3 + 6tu + x = 0. \quad (3)$$

Eq. (3) is the fold singularity which defines the desired asymptotics of solutions.

Suleimanov, JETP **78:5** (1994) 583;
Dubrovin, Comm. Math. Phys. **267** (2006) 117.



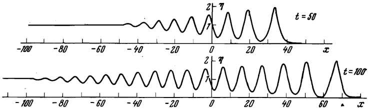
Gurevich, Pitaevskii, JETP **38:2** (1974) 291.

A natural conjecture is that the oscillating zone is described by some solution of (2) with such asymptotics. This is the case, indeed, although the study of this solution is a difficult problem. Even its existence was rigorously proved only in

Claeys, Vanlessen, Nonlinearity **20:5** (2007) 1163.

Decay of initial discontinuity

Another very important Gurevich–Pitaevskii problem is about the evolution of step-like initial data. This problem was analyzed asymptotically by the Whitham averaging method and by some version of the Inverse Scattering Method.



Hruslov, Math. USSR Sb. **28:2** (1976) 229;
Cohen, Comm. PDEs **9:8** (1984) 751;
Kappeler, Diff. Eq. **63:3** (1986) 306;
Khruslov, Kotlyarov, Adv. Sov. Math. **19** (1994) 129.

Gurevich, Pitaevskii,
JETP Lett. **17:5** (1973) 193.

However, this problem does not admit explicit solutions similar to solitons in the rapidly decaying case. It is hardly possible that such a solution can be given by elementary functions or even classical special functions. The conjecture is that an "explicit" answer can be given by the stationary reduction related with the master-symmetry D_{τ_2} [Adler, J. Nonl. Math. Phys. **27:3** (2020) 478] .

Let us describe this equation and its solution in more detail.

Master-symmetry + ...

The general linear combination of symmetries of order ≤ 5 brings to the stationary equation

$$g_2 + k_1 g_1 + k_2 g_0 + k_3 f_2 + k_4 f_1 + k_5 f_0 = 0.$$

The coefficients k_j can be fixed by suitable point transformations, so that this can be simplified to

$$g_2 - 4g_1 = 0.$$

Proposition. KdV equation $u_t = u_3 + 6uu_1$ is consistent with the 6-th order ODE

$$\begin{aligned} 3t(u_4 + 10uu_2 + 5u_1^2 + 10u^3)_x + x(u_2 + 3u^2)_x + 4u_2 + 8u^2 + 2u_1 v \\ - 4(3t(u_2 + 3u^2)_x + xu_1 + 2u) = 0, \quad v_x = u. \end{aligned} \tag{4}$$

It is easy to prove that equation (4) admits two constant solutions:

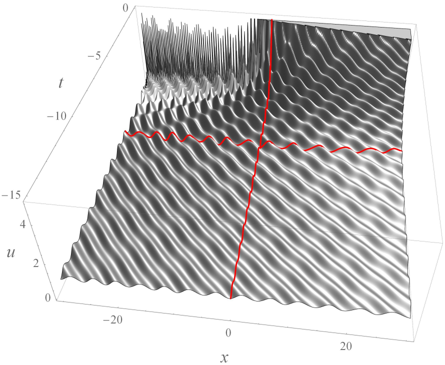
$$u = 0 \quad \text{and} \quad u = 1.$$

The goal is to demonstrate that there exist common solutions of the pair KdV + (4) which are regular for all x, t and has different constant asymptotics for $x \to \pm\infty$ (notice, that there is no much difference between the left and right steps, thanks to the symmetry $x \to -x$, $t \to -t$).

Regularity conditions

A new feature in (4), comparing to the string equations, is that now t is the coefficient at the highest order derivative (it is easy to see that this is always the case when the order of noncommutative component is greater or equal to the order of the commutative one).

As the result, the line $t = 0$ is the fixed singularity of this system, that is a generic solution of the Cauchy problem for this pair of equations is singular along this line.

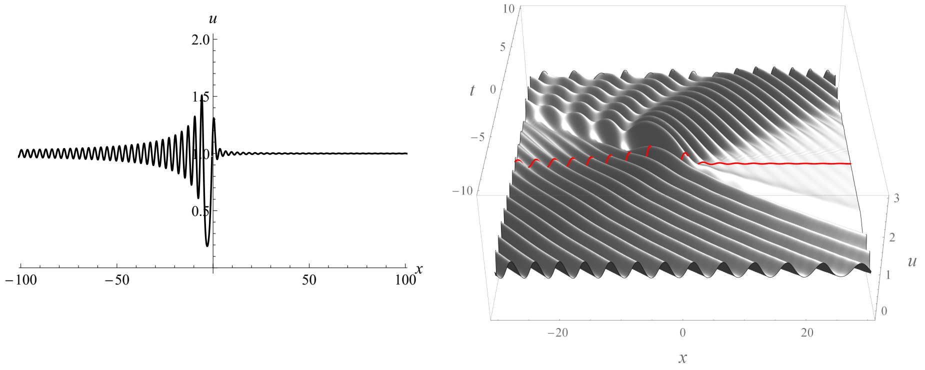


On the other hand, this means that if we require that the solution must be regular then the order of equation is effectively lowered at this line. The Cauchy data at $t = 0$ must satisfy a simpler 4-th order equation

$$x(u_2 + 3u^2)_x + 4u_2 + 8u^2 + 2u_1 v - 4(xu_1 + 2u) = 0, \quad v_x = u. \tag{5}$$

Moreover, the order can be reduced to 2 by use of first integrals (which follow from the Lax representation) and (5) turns out to be equivalent to P_5 equation.

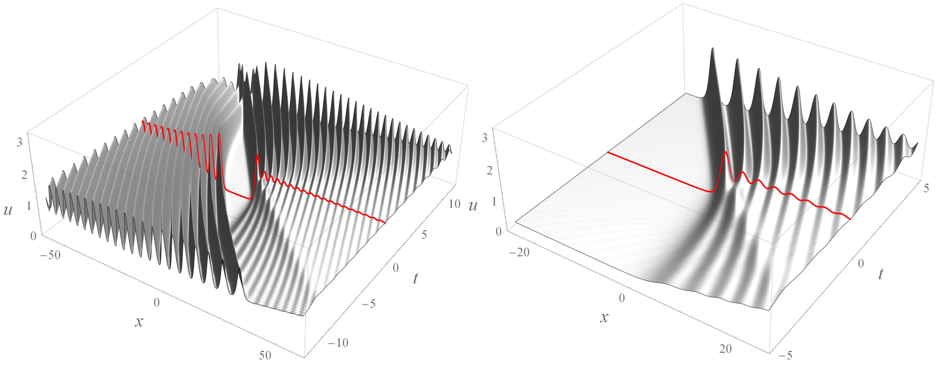
In turn, equation (5) has the fixed singular point $x = 0$. Again, if we require that the solution must be regular then the order of equation is effectively lowered to 1 since the Cauchy data at this point must be constrained. Such solutions can be computed as the Taylor expansions is some finite neighbourhood of the origin and then continued numerically for all real x, t by solving the ODE.



A typical regular solution of (5) has the form of slowly decaying (like x^{-1}) oscillations near $u = 1$, separated by a well near the origin, with different oscillation amplitudes on the left and on the right.

Limiting transition to step-like solutions

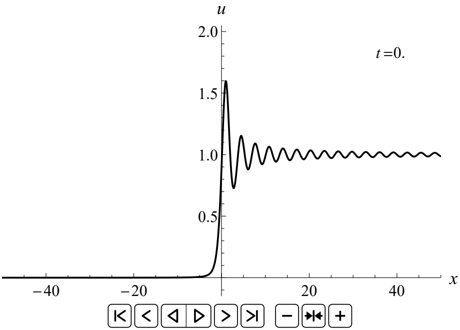
By smoothly varying the initial data, it is possible to broaden the well near the origin. Under this process, the solution remains practically unchanged on one half-line, but changes drastically on the another one. As the result the oscillating zone moves from the origin to infinity. In the limit, we obtain a separatrix step-like solution.



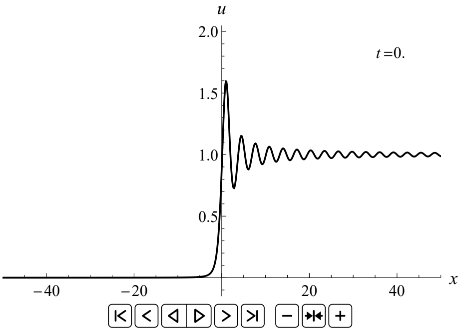
The numerical experiments suggest that equation (4) admits four such step-like solutions for any value of first integrals $H(0) > -4$ and $H(-1) > 0$.

The plots for one of the step-like solutions for $H_0 = -2$, $H_1 = -6$.

compression wave



rarefaction wave



Landau–Lifshitz equation

The hierarchy of the equation

$$s_t = [s, s_{xx} + Js], \quad s \in \mathbb{R}^3, \quad \langle s, s \rangle = 1, \quad J = \mathrm{diag}(J_1, J_2, -J_1 - J_2),$$

can be generated by the local master-symmetry:

$$\begin{aligned}
s_\tau &= x[s, s_{xx} + Js] + [s, s_x], \quad [D_\tau, D_{t_j}] = jD_{t_{j+1}}, \\
s_{t_1} &= s_x, \\
s_{t_2} &= s_t, \\
s_{t_3} &= \Big(s_{xx} + \frac{3}{2}\langle s_x, s_x \rangle s\Big)_x - \frac{3}{2}\langle s, Js \rangle s_x, \\
&\ \ \dots
\end{aligned}$$

moreover, in the (partially) isotropic case there are additional point symmetries of the form $s_{t_0} = As$, $A = -A^t$.

The simplest nonautonomous constraint is of the form

$$2t\Big(s_{xxx} + 3\langle s_x, s_{xx}\rangle s + \frac{3}{2}\langle s_x, s_x\rangle s_x - \frac{3}{2}\langle s, Js\rangle s_x\Big) - x[s, s_{xx} + Js] - [s, s_x] + ks_x = 0. \tag{6}$$

Also the details are very different comparing to the KdV case, the main steps of the construction remains the same.

• (6) is a 6-th order ODE system with 2 first integrals from the Lax representation. The regularity condition at $t = 0$ reduces the order to 2 and we arrive to P_6 equation. The isotropic cases lead to other Painlevé equations as the limiting cases.

• The regularity condition at $x = 0$ distinguishes some special solutions which are defined for all x, t. However, at the moment it is not clear, whether there exist separatrix solutions like the step-like solutions.

• Similar results can be obtained for the NLS equation and other 2-component evolutionary systems. The master-symmetry can be local as in the LL equation or non-local as in the NLS or KdV, but it turns out that this is not very important. In all examples the stationary equation for the master-symmetry is of 6-th order, there are 2 first integrals and the regularity condition also reduces the order by 2.

Volterra lattice

The symmetry algebra of the Volterra lattice (now $u_n \not\equiv \partial_x^n(u)$!!!)

$$u_{n,t} = u_n(u_{n+1} - u_{n-1}), \quad n \in \mathbb{Z},$$

contains the following equations:

$$\begin{aligned}
&u_{n,t_1} = f_1 = u_n(u_{n+1} - u_{n-1}),\\
&u_{n,t_2} = f_2 = u_n(h_{n+1} - h_{n-1}), \quad h_n := u_n(u_{n+1} + u_n + u_{n-1}),\\
&\dots\\
&u_{n,\tau_1} = g_1 = tf_1 + u_n,\\
&u_{n,\tau_2} = g_2 = tf_2 + u_n\big((n + \tfrac{3}{2})u_{n+1} + u_n - (n - \tfrac{3}{2})u_{n-1}\big),\\
&\dots
\end{aligned}$$

The differentiation D_{τ_1} corresponds to the scaling transformation and D_{τ_2} is the master-symmetry. In contrast to the KdV case it is local, but it is not very essential. However, some calculations are simpler than in the KdV case.

Adler, Shabat, JETP Lett. **108:12** (2018) 825;
Theor. Math. Phys. **201:1** (2019) 1442.

Higher symmetry + scaling

- Let us consider the linear combination $u_{n,t_2} + 2u_{n,\tau_0} = 0$.

> Fokas, Its, Kitaev, Russ. Math. Surv. **45:6** (1990) 135;
> Comm. Math. Phys. **142** (1991) 313.

One can easily prove that it is reduced to the 3-point dP_1 equation

$$u_n(u_{n+1} + u_n + u_{n-1}) + 2tu_n + n + (-1)^n b + c = 0 \qquad (\mathrm{dP}_1)$$

with integration constants b, c. This constraint turns the VL into a coupled system for u_{n-1}, u_n which is equivalent to P_4 equation for $y(t) = u_n(t)$:

$$y'' = \frac{(y')^2}{2y} + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{2y}, \qquad (\mathrm{P}_4)$$

$$\alpha = \frac{1}{2}(n - 3(-1)^n b + c), \quad \beta = -(n + (-1)^n b + c)^2.$$

The map $(u_{n-1}, u_n) \mapsto (u_n, u_{n+1})$ is one of the Bäcklund transformations for P_4.

• In order to obtain an analog of the string equation (2) corresponding to the fold singularity, we have to use the next higher symmetry. The stationary equation

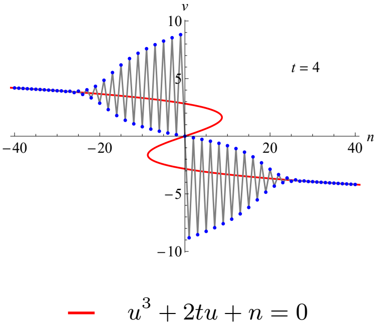
$$u_{n,t_3} + k_2 v_{n,t_2} + k_1 u_{n,t_1} + 2k_0 u_{n,\tau_1} = 0$$

gives, after integration, the 5-point OΔE

$$u_n(h_{n+1} + h_{n-1} + (u_{n+1} + u_n)(u_n + u_{n-1})) \\ + k_2 h_n + k_1 u_n + 2k_0 t u_n + k_0 n + \beta + \gamma(-1)^n = 0.$$

The “dispersionless limit” $u_n = u$ yields the cubic parabola like in (3).

In contrast to the KdV case, a remarkable fact is that for some special choices of parameters this equation admits hypergeometric type solutions. This correspond to the cases when $u_0 = 0$, so that the problem is reduced to the half-line. For instance, if we take $k_2 = k_1 = \beta = \gamma = 0$ and $k_0 = 10$ then the solution is an odd function of n.



Master-symmetry + ...

Let us consider more complicated case

$$u_{n,\tau_2} - 4au_{n,\tau_1} - du_{n,t_1} = 0.$$

Here, a can be scaled either to 0 or to 1 and the shift of t makes possible to omit the term u_{n,t_2}. Like in the [FIK] example, this is a 5-point constraint which can be reduced to a 3-point one, although it is not so obvious.

Proposition. The VL is consistent, for any constants a, b, c, d, with the equation

$$F_n = (q_{n+1} + q_n)(q_n + q_{n-1})u_n - 4(aq_n^2 + (-1)^n bq_n + c) = 0, \tag{7}$$

where

$$q_n := 2tu_n + n - d.$$

In terms of q_n, the constraint (7) turns into the dP_{34} equation while the VL itself turns into a coupled system for u_{n-1}, u_n which is equivalent to the P_5 equation.

• Instead of 4-th order equation in the KdV case we have just 2-nd order for the VL. The regularity condition is also simplified: now it becomes zero order, that is, explicitly solved:

$$u_n(0) = a + \frac{4(-1)^n b(n-d) + 4c + a}{4(n-d)^2 - 1} \quad \text{if} \quad d \notin \frac{1}{2} + \mathbb{Z}$$

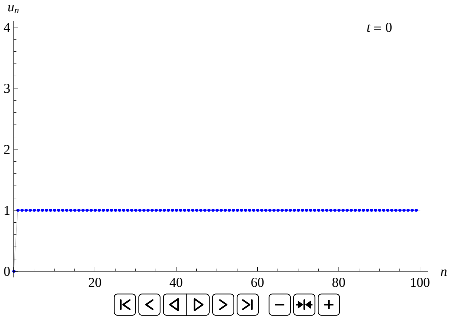
(if d is half-integer then we have to slightly change formula by passing to the limit).

Although this family of admissible initial conditions is very simple, the corresponding solutions are rather interesting. For instance, it contains the step-like solution (for $d = -\frac{1}{2}$) with the initial data

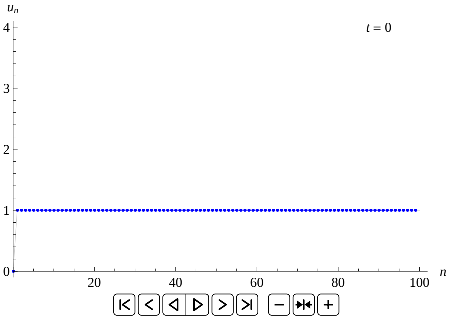
$$u_0 = 0, \quad u_n(0) = 1, \quad n > 0.$$

This should not be understood literally as an analog of the KdV step, this is just a toy model solvable in terms of the hypergeometric functions rather than the Painlevé transcendents. However, the qualitative behaviour is rather similar.

compression wave



rarefaction wave



(In contrast to the KdV, the front of the compression wave is fixed, but we can do this in the KdV by the Galilean transformation.)